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A simple test for quantum channel capacity

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Abstract
Based on state and channel isomorphism we point out that semidefinite programming can be used as a quick test for nonzero one-way quantum channel capacity. This can be achieved by searching for symmetric extensions of states isomorphic to a given quantum channel. With this method we provide examples of quantum channels that can lead to high entanglement transmission but still have zero one-way capacity, in particular, regions of symmetric extendibility for isotropic states in arbitrary dimensions are presented. Further, we derive *a new entanglement parameter* based on (normalized) relative entropy distance to the set of states that have symmetric extensions and show explicitly the symmetric extension of isotropic states being the nearest to singlets in the set of symmetrically extendible states. The suitable regularization of the parameter provides a new upper bound on one-way distillable entanglement.

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1. Introduction

Quantum channels [1] are a very important notion of quantum information theory [2]. It has been proven [1] that there is a connection between entanglement distillation [3] and quantum channel capacities. The no-cloning principle has been used to prove that for some region the quantum depolarizing channel has zero capacity even if does not destroy entanglement [4].

Following seminal work [5] and asymptotic analysis [6] that predicted limit formulae for conjectured hashing inequality [6] recently, the proof of hashing inequality has been provided [7–9]. On the other hand the connection between quantum channels and entanglement distillation [1] has been developed [6, 10] leading in particular to strong nonadditivity effects in the case of more than one receiver [11]. Further, an interesting technique based on approximate quantum cloning was used in [4] to point out a limit of depolarizing a qubit channel. This approach has been further extended in an elegant way to the case of Pauli channels [12] via asymmetric cloning machines.

In the present approach we shall use the above techniques exploiting also a general notion of symmetric extension of a quantum state that was efficiently applied with the help of semidefinite programming to characterize quantum entanglement [13, 14] and states that admit local hidden variables models [15].

To be more specific, in this paper we develop qualitative equivalence between entanglement distillation and quantum channels theory showing in particular that:

(i) semidefinite programming can serve as a simple and quick test for nonzero one-way channel capacity via looking for symmetric extensions of the state $\varrho(\Lambda)$,

(ii) if normalized and regularized, the distance of a given quantum state above to the set of symmetrically extendible states provides *a new entanglement parameter* that leads to an upper bound on one-way distillable entanglement of the state.

To show that SDP can lead to interesting results we provide the family of quantum channels that allow for quite high entanglement transmission but, however, have one-way capacity zero due to the existence of a symmetric extension of the corresponding quantum state. The corresponding extensions are explicitly constructed.

2. One-way distillable entanglement

Following the idea [4] of developing restriction on the qubit depolarizing channel from approximate quantum cloning we shall utilize the general notion of symmetric extensions of quantum states (see [13–15]) to provide a general rule and examples of channels with zero one-way capacity. We show now that every state $\rho_{AB}(\Lambda)$ which has a symmetric extension $\rho_{ABB'}$ has special featured one-way distillable entanglement $D_{\rightarrow}$ and one-way quantum channel capacity $Q_{\rightarrow}$ according to its quantum channel implied by Jamiolkowski isomorphism. The following observation that describes the above reads

Observation 1. If any bipartite state ρ_{AB} has a symmetric extension $\rho_{ABB'}$, so that $\rho_{ABB'} = \rho_{AB'B}$, then for the one-way distillable entanglement there holds: $D_{\rightarrow}(\varrho_{AB}) = 0$.

Proof of the above theorem is immediate and follows from quantum entanglement monogamy (cf [4, 12]). If Alice sends classical information to Bob and they distill singlet in the protocol then the state cannot have symmetric extension since Bob's colleague, say Brigitte (corresponding to index B') could also receive the same message from Alice and finally share the singlet with Alice too. But Alice's particle cannot be maximally entangled with two different particles at the same time (this is just the entanglement monogamy property). So a symmetric extendible state cannot have one-way distillable entanglement nonzero. Combining the above observation we get immediately:

Observation 2. A sufficient condition for one-way quantum capacity of a given quantum channel Λ to vanish is symmetric extendibility of the state $\varrho(\Lambda)$ isomorphic to the channel.

As a special example of application of these observations we use below bipartite state ρ_{AB} that is extendible for $F \leqslant \frac{1}{2}$, moreover, note that in this range the state may be quite strongly entangled

$$\rho_{AB} = \frac{F}{3} P_{+} + \frac{1-F}{3} (|01\rangle\langle 01| + |20\rangle\langle 20| + |21\rangle\langle 21|). \tag{1}$$

Note that filtering on Bob's side the state ρ_{AB} and in general any such a state does not change the extendibility, what may be simply proved. Applying filtering with

$W = \mathrm{diag}\big[1, \frac{1}{\sqrt{F}}, \frac{1}{\sqrt{2-F}}\big]$ we get a state $\widetilde{\rho}_{AB}$ and a maximally mixed state $\widetilde{\rho_A}$ on Alice's side

$$\widetilde{\rho}_{AB} = \frac{W \otimes I\rho_{AB}W^\dagger \otimes I}{Tr\{W \otimes I\rho_{AB}W^\dagger \otimes I\}}, \qquad \widetilde{\rho}_A = \frac{I}{3}$$

$$\widetilde{\rho}_{AB} = \begin{pmatrix} \frac{F}{3} & 0 & 0 & 0 & \frac{\sqrt{F}}{3} & 0 & 0 & 0 & \frac{F}{3\sqrt{2-F}} \\ 0 & \frac{1-F}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{F}}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{\sqrt{F}}{3\sqrt{2-F}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-F}{3(2-F)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1-F}{3(2-F)} & 0 \\ \frac{F}{3\sqrt{2-F}} & 0 & 0 & 0 & \frac{\sqrt{F}}{3\sqrt{2-F}} & 0 & 0 & 0 & \frac{F}{3(2-F)} \end{pmatrix}. \quad (2)$$

For any of the above states the extension can be found by means of linear optimization with the help of SEDUMI module [23]. We have found the extension of ρ_{AB} very easily, in fact we have for $F \leqslant \frac{1}{2}$ the following spectral decomposition of the extension ρ_{BAB}:

$$\begin{cases} |\varphi_0\rangle = |020\rangle \text{ and } \lambda_0 = \dfrac{1-F}{6} \\ |\varphi_1\rangle = |001\rangle + |100\rangle + |111\rangle + |122\rangle + |221\rangle \text{ and } \lambda_1 = \dfrac{F}{3} \\ |\varphi_2\rangle = |021\rangle \text{ and } \lambda_2 = \dfrac{1-2F}{6} \\ |\varphi_3\rangle = |101\rangle \text{ and } \lambda_3 = \dfrac{1-2F}{3} \\ |\varphi_4\rangle = |120\rangle \text{ and } \lambda_4 = \dfrac{1-F}{6} \\ |\varphi_5\rangle = |121\rangle \text{ and } \lambda_5 = \dfrac{1-2F}{6}, \end{cases} \quad (3)$$

where generally eigenvalues have to fulfil the following conditions so that after tracing out Brigitte we obtain ρ_{AB}:

$$\begin{cases} \lambda_0 + \lambda_4 = \dfrac{1-F}{3} \\ \lambda_2 + \lambda_5 = \dfrac{1-2F}{3}. \end{cases} \quad (4)$$

According to these constructions we may find another state ρ_{BAB} that is nearest (in the set of states constructed on the above eigenvectors) to the singlet in the sense of fidelity ($\mathcal{F} = \langle\Psi_+|\rho_{\mathcal{AB}}|\Psi_+\rangle$) of its local reduction ρ_{AB}

$$\begin{cases} \rho_{BAB} = \frac{1}{5}|\varphi_1\rangle\langle\varphi_1| \\ \rho_{AB} = \frac{3}{5}P_+ + \frac{1}{5}|01\rangle\langle 01| + \frac{1}{5}|21\rangle\langle 21|. \end{cases} \quad (5)$$

As a generalization of such states we construct states extreme in the above sense for arbitrary dimension

$$\Upsilon = \frac{d}{2d-1}P_+ + \frac{1}{2d-1}\sum_{i=1}^{d-1}|i0\rangle\langle i0|. \quad (6)$$

We state now the following question as a natural conclusion of the above analysis:

Question: What is the maximal possible value of fidelity of ρ that we may obtain from states for which $Q_{\rightarrow} = 0$?

3. Upper bound on $D_{\rightarrow}$

In this section we consider the distance of any state from the set of extendible states. Note that the set of extendible states is convex and compact which can be obviously obtained from the extendibility of any convex combination of extendible states. Subsequently, we show that the set is closed under local operations and one-way classical communication (1-LOCC) in the following lemma:

Lemma 3.1. *The set $\mathcal{E}_{\mathcal{AB}}$ of symmetrically extendible states is mapped under 1-LOCC for $\Lambda : B(\mathcal{H}_{\mathcal{AB}}) \rightarrow \mathcal{B}(\mathcal{H}_{\widetilde{\mathcal{A}}\widetilde{\mathcal{B}}})$ into the set of symmetrically extendible states $\mathcal{E}_{\widetilde{\mathcal{A}}\widetilde{\mathcal{B}}}$.*

Proof.

$$\begin{aligned}\rho_{AB} \subset \mathcal{E}_{\mathcal{AB}} &\Rightarrow \exists_{\rho_{\mathcal{ABB'}}} \rho_{\mathcal{ABB'}} = \rho_{\mathcal{AB'B}} \wedge Tr_{\mathcal{B'}} \rho_{\mathcal{ABB'}} = \rho_{\mathcal{AB}} \\ &\Rightarrow Tr_{\widetilde{B}'} \Lambda(\rho_{ABB'}) = \rho_{\widetilde{A}\widetilde{B}} \subset \mathcal{E}_{\widetilde{\mathcal{A}}\widetilde{\mathcal{B}}},\end{aligned}$$

where

$$\begin{aligned}\Lambda(\rho_{ABB'}) = \sum_{i,j=1}^{K,L} &\left(I_2^{\widetilde{A}} \otimes W_{ji}^{B\rightarrow\widetilde{B}} \otimes W_{ji}^{B'\rightarrow\widetilde{B}'}\right)\left(V_i^{A\rightarrow\widetilde{A}} \otimes I_1^B \otimes I_1^{B'}\right)\rho_{ABB'} \\ &\times\left(V_i^{A\rightarrow\widetilde{A}\dagger} \otimes I_1^B \otimes I_1^{B'}\right)\left(I_2^{\widetilde{A}} \otimes W_{ji}^{B\rightarrow\widetilde{B}\dagger} \otimes W_{ji}^{B'\rightarrow\widetilde{B}'\dagger}\right)\end{aligned}$$

and operations acting on Bob's side are trace preserving due to the necessity of non-breaking the property of extendibility. □

For our analysis we define the measure of this distance based on the definition of relative entropy

Definition 3.2. *Assume that a convex set $\mathcal{E}_{\mathcal{AB}}$ is a set of extendible states, i.e.*

$$\mathcal{E}_{\mathcal{AB}} = \{\sigma_{\mathcal{AB}} : \exists_{\Psi_{\mathcal{ABB'C}}} \sigma_{\mathcal{AB}} = \sigma_{\mathcal{AB'}} = Tr_{\mathcal{CB}}[|\Psi_{\mathcal{ABB'C}}\rangle\langle\Psi_{\mathcal{ABB'C}}|]\}.$$

Then the distance of a state ρ_{AB} on $\mathcal{H}_{\mathcal{AB}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ with $\dim \mathcal{H}_{\mathcal{A}} = d_{\mathcal{A}}$ and $\dim \mathcal{H}_{\mathcal{B}} = d_{\mathcal{B}}$ from the set of extendible states $\mathcal{E}_{\mathcal{AB}}$ of $d \otimes d$ type where $d = \max[d_A, d_B]$ is defined by

$$R_{\mathcal{E}_{\mathcal{AB}}}(\rho_{AB}) = \delta_{AB} \inf_{\sigma_{AB} \in \mathcal{E}} R(\widetilde{\rho}_{AB} \| \sigma_{AB}), \tag{7}$$

where $\forall_{\rho,\sigma} R(\rho\|\sigma) = Tr[\rho \log \rho - \rho \log \sigma]$ and $\delta = -\frac{\log d}{\log \frac{(d+1)}{2d}}$ with $d = \max[d_A, d_B]$ due to normalization of this function on maximally entangled states. In formula (7) $\widetilde{\rho}_{AB}$ is taken as a state of $d \otimes d$ type (after embedding ρ_{AB} into $d \otimes d$ space).

Using techniques [26] we show that the nearest one in an arbitrary dimension is a state $\rho(d, F_{\max})$ from a subset of isotropic states $\rho(d, F)$ [27] with fidelity $F \leqslant F_{\max}$ for which those are symmetrically extendible

$$F_{\max} = \frac{d+1}{2d} \tag{8}$$

$$\rho(d, F) = \frac{d^2}{d^2 - 1}\left[(1 - F)\frac{I}{d^2} + \left(F - \frac{1}{d^2}\right)P_+\right]. \tag{9}$$

Indeed, following [26] one needs to analyze operators from a six-dimensional non-commutative $\mathbf{C}^*$-algebra that are $\overline{U} \otimes U \otimes U$-invariant and $V_{(23)}$-invariant. Such operators S will be represented as a linear combination of the basis elements of the algebra: $B = \{S_+, S_-, S_0, S_1, S_2, S_3\}$ where for the trace condition one obtains [26] conditions for factors of the combination: $s_2 = s_3 = 0$ and, further, from positivity: $s_0 = 1 - s_+ - s_-$.

$$S = s_+ S_+ + s_- S_- + s_0 S_0 + s_1 S_1. \tag{10}$$

The matter of interest is now the tetrahedron in three-dimensional Euclidian space of parameters (s_+, s_-, s_1) confined by the hyperplanes [26]: $\{h_1^{'}, h_2^{'}, h_3^{'}, h_4^{'}\}$ in which exists the state Ω_{ABE} giving the searched symmetric extendible reduction ρ_{AB}. For maximizing the distance of the unknown state ρ_{AB} to singlet it suffices to find the maximization over fidelity $\widetilde{F}$ between the symmetric extension represented as Ω_{ABE} and virtually extended unnormalized operator $\rho_{ABB'} = P_+ \otimes I$ as $\widetilde{F}_{\max} = Tr[P_+ \otimes I \Omega_{ABE}] = Tr[P_+ \rho_{AB}] = F_{\max}$:

$$\begin{cases} F_+ = Tr[(P_+ \otimes I)S_+]/Tr\left[S_+^2\right] = 0 \\ F_- = Tr[(P_+ \otimes I)S_-] = 0 \\ F_0 = Tr[(P_+ \otimes I)S_0]/Tr\left[S_0^2\right] = d/2d \\ F_1 = Tr[(P_+ \otimes I)S_1]/Tr\left[S_1^2\right] = 1/2d \end{cases} \tag{11}$$

$$\begin{cases} \widetilde{F} = F_0 + \overrightarrow{s} \circ \overrightarrow{f} \\ \widetilde{F}_{\max} = \max_{\overrightarrow{s} \in \Delta} \widetilde{F}, \end{cases} \tag{12}$$

where Δ denotes the tetrahedron bounded by mentioned hyperplanes, $\overrightarrow{f} = [F_+ - F_0, F_- - F_0, F_1]$ and $\overrightarrow{s} = [s_+, s_-, s_0]$. Normalization of parameters F_i inherits from the commutation relations [26] between operators S_i. Maximization results in $\overrightarrow{s} = [0, 0, 1]$ that relates to the found aforementioned isotropic states $\rho_{AB} = \rho(d, F_{\max})$. The explicit form of the tripartite symmetric extension of isotropic states $\rho(d, F_{\max})$ in the border of extendibility is

$$\Omega_{ABE} = \frac{1}{2d}(S_0 + S_1), \tag{13}$$

where [26]

$$\begin{cases} S_0 = \dfrac{1}{d^2 - 1}(d(X + VXV) - (XV + VX)) \\ S_1 = \dfrac{1}{d^2 - 1}(d(XV + VX) - (X + VXV)) \end{cases}$$

for

$$|\Phi\rangle = \sum_i |ii\rangle, \qquad X = |\Phi\rangle\langle\Phi| \otimes I, \qquad V = V_{(23)} = \sum_{ijk} |ijk\rangle\langle ikj|.$$

It is important to note that the same results can be obtained numerically by means of linear programming methods that we have utilized to find the broad class of symmetric extendible states.

Following this, we analyze if similarly to distance from separable states one can construct an appropriate entanglement measure based on (7). The normalized distance from the set of extendible states does not satisfy though all necessary conditions [24, 25] that every measure of one-way distillable entanglement has to satisfy; introduction of the normalization factor δ_{AB} causes $R_{\mathcal{E}_{AB}}(\rho)$ to become explicitly dependant on the dimension of the system AB, therefore, for protocols increasing dimension of the input state the parameter is not a monotone

(A1) If σ_{AB} is separable then $R_{\mathcal{E}_{AB}}(\sigma_{AB}) = 0$ due to the fact that every separable state is extendible.

(A2) Local unitary operations leave $R_{\mathcal{E}_{AB}}(\sigma_{AB})$ invariant that is satisfied due to invariancy of distance measures under local unitary transformations, i.e. $R_{\mathcal{E}_{AB}}(\sigma_{AB}) = R_{\mathcal{E}_{AB}}\big(U_A \otimes U_B\sigma_{AB}U_A^{\dagger} \otimes U_B^{\dagger}\big)$.

(A3) (Restricted 1-LOCC monotonicity.) The parameter $R_{\mathcal{E}_{AB}}(\sigma_{AB})$ of one-way distillable entanglement does not increase under non-increasing dimension 1-LOCC, i.e. $\Lambda : B(\mathcal{H}_{AB}) \to \mathcal{B}(\mathcal{H}_{\widetilde{A}\widetilde{B}})$ with $n_{AB} = \max[d_A, d_B], n_{\widetilde{A}\widetilde{B}} = \max[d_{\widetilde{A}}, d_{\widetilde{B}}]$ for $n_{AB} \geqslant n_{\widetilde{A}\widetilde{B}}$, then

$$R_{\mathcal{E}_{\widetilde{A}\widetilde{B}}}(\Lambda\sigma_{AB}) \leqslant R_{\mathcal{E}_{AB}}(\sigma_{AB}). \tag{14}$$

This condition may be simply proved due to non-increasing of $R(\rho\|\sigma)$ under a subclass of 1-LOCC operations Λ that is stated above in the lemma. Namely, because $\Lambda\mathcal{E}_{AB} \subset \mathcal{E}_{\widetilde{A}\widetilde{B}}$ and assuming that σ^* is an extendible state that realizes the minimal value in equation (7) we have

$$\begin{aligned} R_{\mathcal{E}_{AB}}(\rho) &= \delta_{AB}R(\rho\|\sigma^*) \geqslant \delta_{\widetilde{A}\widetilde{B}}R(\Lambda\rho\|\Lambda\sigma^*) \\ &\geqslant \delta_{\widetilde{A}\widetilde{B}} \inf_{\sigma\in\mathcal{E}_{\widetilde{A}\widetilde{B}}} R(\Lambda\rho\|\sigma) = R_{\mathcal{E}_{\widetilde{A}\widetilde{B}}}(\Lambda\rho), \end{aligned}$$

where $n_{AB} \geqslant n_{\widetilde{A}\widetilde{B}}$ derives the condition $\delta_{AB} \geqslant \delta_{\widetilde{A}\widetilde{B}}$.

However, we show further that the entanglement parameter can be utilized for bounding one-way entanglement of distillation due to the preparation of the measure in the asymptotic regime.

In general, every entanglement parameter of type $E(\sigma) = \alpha \inf_{\rho\in\Delta} \mathcal{D}(\sigma\|\rho)$ where $\mathcal{D}(\sigma\|\rho)$ is appropriate distance between σ and ρ, Δ denotes the characteristic set to which the distance is measured and α normalizes the parameter so that $E(|\Psi_+\rangle\langle\Psi_+|) = \log d$ is not monotonic, i.e. $\exists_\Lambda E(\sigma) > E(\Lambda(\sigma))$. For $R_{\mathcal{E}_{AB}}$ unitary injection of input state ρ_{AB} into higher dimensional space gives $R_{\mathcal{E}_{AB}}(\rho) > R_{\mathcal{E}_{\widetilde{A}\widetilde{B}}}(\Lambda(\rho))$.

Additionally, following analysis in [28, 29], we show that the entanglement parameter satisfies:

B1. (Continuity on isotropic states.) We may simply show that this parameter is continuous on isotropic states $\rho(d_n, F_n)$ with $F_n \to 1, d_n \to \infty$ that means

$$\frac{R_{\mathcal{E}}(\rho(d_n, F_n))}{\log d_n} \to 1$$

as then $R_{\mathcal{E}}(\rho(d_n, F_n)) \to \log d_n$ that is easy to check.

Following the papers [28, 30] and the above definition we define the distance in the asymptotic regime as follows:

$$R^{\infty}_{\mathcal{E}_{AB}}(\rho_{AB}) = \limsup_{n\to\infty} \frac{R_{\mathcal{E}_{AB}}(\rho_{AB}{}^{\otimes n})}{n}. \tag{15}$$

One can also propose other measures, but this will be the subject of analysis elsewhere[3].

Having defined the above regularized parameter $R^{\infty}_{\mathcal{E}_{AB}}(\rho_{AB})$, we are now able to determine an upper bound on the one-way distillable entanglement. In [9] Devetak and Winter have proved a very powerful conjecture called 'hashing inequality'

$$D_{\to} \geqslant S(\rho_B) - S(\rho_{AB})$$

[3] For instance one can propose fidelity of state according to the nearest purified extension as follows $F_{\mathcal{E}}(\rho_{AB}) = \inf_{\sigma_{AB}\in\mathcal{E}} F(\rho_{AB}, \sigma_{AB}) = \inf_{\sigma_{AB}\in\mathcal{E}} |\langle\Phi_{ABB'C}|\Psi_{ABB'C}\rangle|$ where the set $\mathcal{E}$ is defined as in the above definition and both $\Phi_{ABB'C}$ and $\Psi_{ABB'C}$ are a purification of a suitable state. It can be shown that such quantity is also restricted 1-LOCC monotone.

from which one may find particular states of nonzero $D_{\rightarrow}$. For the very features of measures that bound the distillable entanglement $D_{\rightarrow}$, defined in [28, 29], where it was shown that monotonicity and continuity on isotropic states are sufficient for any properly regularized function to be upper bound for $D_{\rightarrow}$, we may prove now the following theorem exploiting only distillation protocols in the line of the proof:

Theorem 3.3. *For any bipartite state ρ_{AB} there holds:*

$$D_{\rightarrow}(\rho_{AB}) \leqslant R^{\infty}_{\mathcal{E}_{AB}}(\rho_{AB}). \tag{16}$$

Proof. Any one-way distillation protocol can be reduced to the distillation protocol [28–30] where the input is $\rho^{\otimes n}$ and the output is a family of the states $\rho(d_n, F_n)$ with $\lim_{n\rightarrow\infty} \frac{\log d_n}{n} = D_{\rightarrow}(\rho)$ and $F_n \rightarrow 1$. We may always put $d_n \leqslant n^n_{AB}$ for $n_{AB} = \min[d_A, d_B]$ since there holds $D_{\rightarrow}(\rho) \leqslant \log n_{AB}$. Thus, we can consider only 1-LOCC non-increasing dimensions of input and so monotonicity of $R_{\mathcal{E}_{AB}}$ holds. By analogy with the theorem put in [28–30] the properties (A3) and (B1) imply that $R^{\infty}_{\mathcal{E}_{AB}}(\rho_{AB})$ is an upper bound for $D_{\rightarrow}$. The regularization (15) with supreme value enables the upper bound of $D_{\rightarrow}$. □

4. Conclusions

Quantum channels theory still has many unsolved problems. We have pointed out a general test for zero capacity of one-way channel capacity (which has been shown to be equal to zero-way capacity [18]). The test is based on checking the existence of a symmetric extension of a state isomorphic to a given channel. The test can be very easily performed with the help of popular semi-definite programming codes. Finally, based on the test, we have found a new parameter of entanglement. Its suitable regularized version is an upper bound on one-way distillable entanglement of a given quantum state. Note that, although the entanglement monogamy property has been known for a long time, this is the first entanglement parameter based explicitly on that property and symmetric extendibility of quantum states. We hope that the above results will help in further analysis of various aspects of quantum channels. It is very interesting that the recently developed complete hierarchies approach to the separability problem [33] has been extended [32] to include symmetric extensions of quantum operators which leads to class of entanglement measures. This gives hope that symmetric extensions will be a useful tool not only to qualify but also to quantify some aspects of quantum entanglement.

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